# Julius Caesar and Basic Law V*

*Richard Kimberly Heck*

Much of Frege's philosophical and mathematical work is devoted to an attempt to show that, given appropriate definitions, all theorems of arithmetic can be proven from logical laws alone. In his *Grundgesetze der Arithmetik*, Frege presents formal proofs intended to show "that arithmetic is a branch of logic and need not borrow any ground of proof whatever from either experience or intuition" (Frege, 1962, v. I, p. 1). But the formal system in which Frege proves the basic laws of arithmetic is inconsistent, since Russell's Paradox is derivable from Frege's Basic Law V in (full) second-order logic.[1] Basic Law V is:

$$(\grave{\epsilon}(F\epsilon) = \grave{\epsilon}(G\epsilon)) \ = \ \forall x(Fx = Gx)$$

Law V governs the term forming operator "$\grave{\epsilon}(\phi\epsilon)$", from which terms standing for 'value-ranges' are formed: It states that the value-range of $F\xi$ is the same as that of $G\xi$ just in case $F$ and $G$ have the same values for the same arguments. Since, for Frege, the truth-values are objects, extensions of concepts are among the value-ranges.

Value-ranges are used throughout Part II of *Grundgesetze*, in which Frege proves the axioms of arithmetic and various related results. As argued in Chapter **??**, however, with just two exceptions, Frege uses value-ranges only for convenience, to make certain parts of his proofs easier; most of his uses of them can be eliminated in a uniform manner. The two ineliminable uses occur in Frege's proofs of the two directions of HP: The number of objects falling under a concept $F\xi$ is the same as the number of objects falling under $G\xi$ if, and only if, the $F$s can be correlated one-to-one with the $G$s.

HP is stated by Frege in *Die Grundlagen*, and he there uses it in informal proofs of various fundamental facts about the natural numbers, including axioms for arithmetic. Famously, however, Frege derives HP, in *Die Grundlagen*, from an explicit definition of numbers as extensions of concepts—much

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[1] In fact, what one needs for the derivation is just $\Sigma^1_1$ comprehension (Heck, 1996).

as he derives it, in *Grundgesetze*, from a definition of numbers as value-ranges. But once again, although the use of extensions is necessary for the proof of HP from the explicit definition, Frege makes no essential use of extensions during the derivation of the axioms of arithmetic *from* HP. His proof-sketches therefore amount to an informal derivation of the laws of arithmetic from HP alone. Similarly, since, in *Grundgesetze*, Basic Law V is used essentially only in the proof of HP, the proofs Frege there gives of the axioms of arithmetic amount to a formal second-order derivation of them from HP (*modulo* the inessential uses of Basic Law V).

As Frege Arithmetic—second-order logic, with HP taken as the sole 'non-logical' axiom—is equi-interpretable with second-order arithmetic (Boolos and Heck, 1998, appendix 2), it follows that Frege's formal proof of the axioms of arithmetic can be carried out within a (presumably) consistent sub-theory of the formal theory of *Grundgesetze*. Furthermore, Frege knew full well that the other uses he made of value-ranges were only for convenience (see Section **??**). The point of our discussion up to this point is thus this: Frege knew that the basic laws of arithmetic could be derived from HP in that sub-system of his formal system that results from the exclusion of Basic Law V.[2] Formally speaking, then, there was no reason that, upon receiving Russell's famous letter, Frege could not have abandoned Law V, installed HP as an axiom, eliminated the inessential uses of value-ranges, and then have declared himself to have derived the axioms of arithmetic from HP, a principle arguably "analytic of the concept of number", as neo-Fregeans might put it. That would have been no mean feat.

Moreover, not only did Frege know that he could have substituted HP for Basic Law V, he explicitly considered doing so. In a letter written to Russell in 1902, discussing how he might avoid using Law V, Frege writes:

> We can also try the following expedient, and I hinted at this in my *Foundations of Arithmetic*. If we have a relation $\Phi(\xi, \eta)$ for which the following propositions hold: (1) from $\Phi(a, b)$ we can infer $\Phi(b, a)$, and (2) from $\Phi(a, b)$ and $\Phi(b, c)$ we can infer $\Phi(a, c)$; then this relation can be transformed into an equality (identity), and $\Phi(a, b)$ can be replaced by writing, e.g., "$\S a = \S b$". If the relation is, e.g., that of geometrical similarity, then "$a$ is

[2] And, strictly speaking, Basic Law VI, which governs the description-operator and is formulated in terms of value-ranges. Frege uses Basic Law VI only in his definition of the application-operator. It is therefore not needed once value-ranges are excluded from the system. The system remaining once Axioms V and VI are dropped is a version of axiomatic second-order logic, with comprehension formulated as a rule of substitution, for which see Frege's Rule 9 (Frege, 1962, v. I, §48).

> similar to $b$" can be replaced by saying "the shape of $a$ is the same as the shape of $b$". This is perhaps what you call "definition by abstraction". But the difficulties here are [][3] the same as in transforming the generality of an identity into an identity of value-ranges. (Frege, 1980b, p. 141)

The idea is indeed familiar from *Die Grundlagen*: If $\Phi(\xi, \eta)$ is an equivalence relation, we may take

$$\mathrm{fnc}(a) = \mathrm{fnc}(b) \;\equiv\; \Phi(a, b)$$

as a 'contextual definition' of the functional expression "$\mathrm{fnc}(\xi)$" (as an axiom governing it, in the formal system).[4] HP, of course, is a somewhat different case: The relation of equinumerosity, though provably an equivalence relation, is one between *concepts*, not objects, so HP is a second-order abstraction principle. Writing "$\mathrm{Eq}_x(\Phi x, \Psi x)$" for any of the usual formalizations of "The $\Phi$s can be correlated one-one with the $\Psi$s", and "$\mathrm{N}x : \Phi x$" for "the number of $\Phi$s", HP may then be formulated as follows:

$$\mathrm{N}x : Fx = \mathrm{N}x : Gx \;\equiv\; \mathrm{Eq}_x(Fx, Gx)$$

Thus, to adopt HP as a fundamental axiom is precisely to follow the suggestion Frege is making in his letter to Russell.

Our question is why Frege did not take his own advice: Abandon Basic Law V, install HP as an axiom, and make one's stand on the logical character of HP itself.

## 1 The Caesar Problem

In the cited letter to Russell, Frege remarks that there are certain difficulties connected with adopting HP as a primitive axiom, ones that are, in fact, the same as certain difficulties that afflict Basic Law V. I take it that Frege is not suggesting that HP is inconsistent.[5] What difficulties might he have in

[3] At this point, the translation contains the word "not", which is not found in the German edition. Thanks to Michael Kremer for originally pointing this out to me and to Thorsten Sander for reminding me again, more recently. Christian Thiel has confirmed that the German edition is faithful to Frege's original letter.

[4] Frege omits the condition of reflexivity—$\forall x[(\exists y)(Rxy \vee Ryx) \rightarrow Rxx)$—but the other two conditions imply it, though they do not imply the stronger condition that $\Phi$ is *totally* reflexive—$\forall x(Rxx)$.

[5] Indeed, since Frege is talking about abstraction principles quite generally, such a suggestion would be absurd.

mind, then? He does not say explicitly, but it is natural to look for them in *Die Grundlagen*. Frege there discusses, at some length, the question whether a principle similar in spirit to HP can be taken as explaining the concept of direction. In this case, the principle is:

$$\mathrm{dir}(a) = \mathrm{dir}(b) \;\equiv\; a \parallel b$$

That is: The direction of $a$ is the same as the direction of $b$ if, and only if, $a$ is parallel to $b$. Frege considers three objections to the claim that this principle explains the concept of direction, the first two of which he rebuts. In the end, though, he rejects the proposed explanation on the ground that it fails to decide the truth-values of what have come to be called *mixed identity-statements*, identity statements of the form "$t = \mathrm{dir}(a)$", where $t$ is a term not itself of the form "$\mathrm{dir}(x)$". Frege's own example is "England is the direction of the Earth's axis" (Frege, 1980a, §66). It is this problem, the so-called Caesar problem,[6] that prevents Frege from regarding HP as explaining the concept of number—and so, within the formal theory, from adopting it as an axiom.

It is important to realize that the Caesar problem itself is *not* one upon whose solution the *formal* part of the logicist project depended—and that Frege knew as much. Frege knew that the axioms of arithmetic are derivable from HP, and it is simply obvious that the derivation does not require a solution to the Caesar problem.

Frege raises the Caesar problem as an objection to his 'contextual' explanation of the concept of direction, an explanation that is offered as part of an attempt to answer the famous question of *Die Grundlagen* §62: "How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them?" According to Frege, to answer this question, it is necessary (and apparently sufficient) to explain the senses of identity statements in which number-words occur; analogously, it is necessary (and apparently sufficient), in order to answer the question how directions are given to us, to explain the senses of identity statement in which names of directions occur. The suggestion Frege is considering, when he raises the Caesar problem, is that this explanation may be given by means of the abstraction principle considered above; the analogous suggestion, in the case of numbers, is that the senses of identity statements containing names of numbers may be explained by means of HP.

[6] So-called because Frege argues, a little earlier, that a familiar sort of inductive definition of names of finite numbers fails to decide whether Caesar is a number (Frege, 1980a, §56). I discuss the relation between these two versions of the Caesar problem in Chapter ??.

It is far from obvious, however, on what ground Frege concludes, from the failure of the relevant abstraction principle to decide whether England is a direction, that it fails as an explanation of the senses of identity statements containing names of directions. Of course, the Caesar problem does show that the abstraction principle, on its own, does not provide a sense for *all* identity statements containing names of directions, since it does not provide one for "England is the direction of the Earth's axis".[7] But why should that be thought a difficulty? We shall return to that question. At present, the important point is just that extensions of concepts—and later, value-ranges—are introduced by Frege in order to resolve the Caesar problem (Frege, 1980a, §68). Rather than attempt to explain the senses of identity statements involving names of numbers by means of HP, Frege explicitly defines the number of $F$s as the extension of the concept "concept which can be correlated one-one with the concept $F$", "assum[ing] that it is known what the extension of a concept is" (Frege, 1980a, §68, note). Frege then derives HP from this explicit definition and, as said, proves the axioms of arithmetic from HP, making no further use of extensions (nor any essential use of value-ranges).

As has been said, Frege knew that he could do without extensions *formally*. His abandonment of the logicist program is thus, in a certain sense, not the result of Russell's discovery of the contradiction. Russell of course showed Basic Law V to be inconsistent, but this axiom plays a very limited role in Frege's proofs. What ultimately forces Frege to abandon his logicism is his inability to resolve the Caesar problem, his inability, without making reference to value-ranges, to answer the question how we apprehend[8] logical objects. Indeed, just before mentioning to Russell that HP might replace Law V, and alluding to the "difficulties" confronting this suggestion, he writes:

> I myself was long reluctant to recognize the existence of value-ranges and hence classes; but I saw no other possibility of placing arithmetic on a logical foundation. But the question is, How do we apprehend logical objects? And I have found no other answer

[7] It is worth emphasizing that this is the objection. The objection is not just that the contextual definition fails to decide the *truth-value* of this sentence, but that it fails to give any clear sense to it at all.

[8] I shall use this term of Frege's throughout, though it obviously could use some explanation. In my own opinion, Frege is using the term to mean "refer to" or "think about", that is, "have cognitive access to". What is wanted is thus an account of how we might be able to refer to or think about certain objects though we have neither intuition nor experience of them.

> to it than this, We apprehend them as extensions of concepts, or more generally, as value-ranges of functions. I have always been aware that there were difficulties with this, and your discovery of the contradiction has added to them; but what other way is there? (Frege, 1980b, pp. 140–1)

The question how we apprehend logical objects is much the same question as that raised in §62 of *Die Grundlagen*: For logical objects are those our apprehension of which does not depend upon intuition or experience either of them or of objects by means of which they are identified.[9] Thus: Frege introduces extensions of concepts into his system to explain how we apprehend logical objects. It was because he could not otherwise explain how we apprehend logical objects that he could not do without extensions.

Frege's abandonment of the logicist program is thus the result of a failure to resolve not a *formal* problem but an *epistemological* one. If we are to understand Frege's logicism, we therefore must understand, first, what Frege meant by the question how we apprehend logical objects and, second, why the Caesar problem frustrates the attempt to answer this question by means of abstraction principles, such as HP.

## 2 The Caesar Problem in *Grundgesetze*

I have argued that Frege is unwilling to adopt HP as a fundamental axiom because he does not think he can solve the Caesar problem without making reference to value-ranges. In this section, we shall look at some of Frege's later discussions of the Caesar problem: It continued to haunt him long after he 'solved' it in *Die Grundlagen*. We shall see that, in fact, Frege was never able to solve the Caesar problem to his satisfaction.

Frege's 'solution' of the Caesar problem in *Die Grundlagen* consists in the identification of numbers with the extensions of certain concepts. As has often been remarked, however, this solution works only if we assume that we know how to resolve the Caesar problem for extensions themselves: The identification of numbers with extensions decides whether Caesar is a number only if it has already been decided whether Caesar is an extension and, if so,

[9] Care is needed here. It is tempting to say just that logical objects are those an apprehension of which requires neither intuitions nor ideas of them. As said below, however, directions are not *logical* objects, but their apprehension does not rest upon intuition or experience *of directions* (Frege, 1980a, §64). Nonetheless, Frege would maintain that apprehension of directions does require intuition, since one must apprehend the direction as the direction of a given line. The question raised in §62 is thus more general than the question how we apprehend logical objects.

which one he is. Frege's remark that, in giving his solution, he is "assum[ing] it is known what the extension of a concept is" (Frege, 1980a, §68, note; see also §107) is naturally interpreted as a recognition of this fact. The Caesar problem is, therefore, not so much solved by Frege's identification of numbers with extensions as it is relocated by it.

In *Grundgesetze*, reference to extensions is formalized as reference to value-ranges, and value-range terms are governed by Basic Law V, which bears a marked formal similarity to HP. It is therefore not surprising that Frege should raise the question, in §10 of *Grundgesetze*, whether either of the truth-values (Truth and Falsity) is a value-range and, if so, which value-ranges they are. Frege argues that, consistently with Basic Law V, Truth and Falsity may be identified with the value-ranges of any (extensionally distinct) functions. Frege chooses to identify each of them with its own unit class.

Thus, something much like the Caesar problem arises in *Grundgesetze*, and Frege resolves it by making a stipulation regarding the references of certain terms. Now, the domain of Frege's theory consists only of Truth, Falsity, and the value-ranges:[10] So, for whatever *formal* purposes Frege might have needed to resolve the Caesar problem—for whatever reason he might need to fix the truth-values of mixed identity-statements *of the formalism*—his stipulation may suffice. Nevertheless, the Caesar problem, as it is raised in *Die Grundlagen*, is surely not a problem Frege was prepared to resolve by a stipulation applicable only to such objects as are in the domain of the formal theory. A similar 'stipulative' solution would work just as well in the context of second-order logic augmented by HP: Identify Truth with 1; Falsity, with 0.

In a long footnote, Frege considers the question whether a general solution to the Caesar problem can be modeled upon the partial solution he offers in the case of the truth-values: "A natural suggestion is to generalize our stipulation so that every object is regarded as a value-range, viz., as the extension of a concept under which it and it alone falls" (Frege, 1962, v. I, §10). Frege's argument against this proposal is that it works only for such objects as are not "already given to us as value-ranges". Consider $\grave{\alpha}(F\alpha)$, the value-range of the concept $F\xi$. Law V implies that the unit class of $\grave{\alpha}(F\alpha)$, i.e., $\grave{\epsilon}(\epsilon = \grave{\alpha}(F\alpha))$, is the same as $\grave{\alpha}(F\alpha)$ just in case the objects falling under

[10] This claim contradicts the oft-made claim that, as a consequence of his so-called 'universalism', the domain of Frege's theory must always comprise all the objects there are. For further discussion of this matter, and of other issues related to those discussed in this paragraph, see my paper "*Grundgesetze der Arithmetik* I §10" (Heck, 1999).

$\xi = \grave{\alpha}(F\alpha)$ are exactly the objects falling under $F\xi$, i.e., that:

$$\grave{\alpha}(F\alpha) = \grave{\epsilon}(\epsilon = \grave{\alpha}(F\alpha)) \equiv \forall x(Fx \equiv x = \grave{\alpha}(F\alpha))$$

As Frege says, however, "Since this... is not necessary, our stipulation cannot remain intact in its general form": Not *every* object can be the same as its unit class; in particular, no class that does not have exactly one member can be its own unit class. The most natural emendation would be this: Objects *other than* value-ranges are to be the same as their unit classes. But that does not work either: If $x$ were not a value-range, the stipulation would imply that it *was* a value-range, namely, its own unit class, whence it ought *not* to be identified with its own unit class. (A thing cannot both be and not be a value-range.) So we are forced to say instead that every object which is not *obviously* a value-range, which is not "already given to us as a value-range", is to be identified with its unit class. Thus intrude ways in which objects are given.

Frege's discussion of this proposal is reminiscent of his discussion in *Die Grundlagen* of the suggestion that Caesar is not a number because only such objects are numbers as are "introduced by means of" HP:[11]

> If... we were to adopt this way out, we should have to be presupposing that an object can be given only in one single way; for otherwise it would not follow, from the fact that [an object] *was* not introduced by means of our definition, that it *could* not have been introduced by means of it. (Frege, 1980a, §67)

The proposal under consideration in the footnote in section 10 of *Grundgesetze* is rejected on similar grounds:

> ... [I]t is intolerable to allow [the stipulation] to hold only for such objects as are not given us as value-ranges; the way in which an object is given is not an immutable property of it, since the same object can be given in a different way.

Thus, while Frege will make stipulations regarding the truth-values of certain identity statements, for whatever formal reason he might need to do so, he twice rejected attempts to model a *general* solution to the Caesar problem on such stipulations.[12]

[11] Frege is actually discussing directions here, but of course the discussion is meant to apply equally to the case of numbers.

[12] Frege discusses such 'stipulative' solutions to the Caesar problem in at least one other

Frege thus had no general solution to the Caesar problem, there being no other proposed solutions he considers. If so, however, the Caesar problem posed an enormous threat to his position. According to Frege, the Caesar problem shows that the question how we apprehend numbers as logical objects cannot be answered by means of abstraction principles such as HP. Similarly, the analogue of the Caesar problem, for value-ranges, ought to show that our apprehension of value-ranges as logical objects cannot be explained in terms of Basic Law V alone. But the only thing Frege has to say about what value-ranges are is this:

> I use the words "the function $\Phi(\xi)$ has the same *value-range* as the function $\Psi(\xi)$" generally to denote the same as the words "the functions $\Phi(\xi)$ and $\Psi(\xi)$ have always the same value for the same argument". (Frege, 1962, v. I, §3)

And that amounts to explaining what value-ranges are by means of a metalinguistic version of Basic Law V. It follows that, since he was without a solution to the Caesar problem for value-ranges, Frege cannot explain, even to his own satisfaction, how we can apprehend *value-ranges* as logical objects. Since his view was that we apprehend all logical objects *as* value-ranges, he was therefore unable to explain, to his own satisfaction, how we can apprehend logical objects at all.

We have yet, however, to see just why Frege thought the Caesar problem such a threat, for we have yet to see why he thought it showed abstraction principles on their own to be explanatorily impotent.

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place. In one of his letters, Russell had expressed concern about the inference from "The members of $u$ are the same as the members of $v$" to "$u$ is the same as $v$". The inference holds, in Frege's system, only when both $u$ and $v$ are value-ranges: Non-value-ranges have no members, so all non-value-ranges have the same members. Russell therefore asks Frege how it can be known whether a given object is a value-range (Frege, 1980b, p. 139). Unsurprisingly, Frege is unable to answer Russell's question in the general terms in which it is posed. So he says instead that the question may be answered piecemeal, for each sort of mathematical object, as it is introduced:

> Now, all objects of arithmetic are introduced as value-ranges. Whenever a new object is to be considered which is *not introduced as a value-range*, we must at once answer the question whether it is a value-range, and the answer is probably always no, since it would have been introduced as a value-range if it was one. (Frege, 1980b, p. 142, emphasis added)

Similarly, Frege writes in section 10 of *Grundgesetze* that we shall have to decide such questions as they arise, and that "... this can then be regarded... as a further determination of the value-ranges...". Plainly, Frege is not here offering a solution to the Caesar problem: A piecemeal 'solution' is not a solution to the problem but a recipe for side-stepping it.

## 3 The Caesar Problem and the Apprehension of Logical Objects

In the first section, I argued that what prevented Frege from adopting HP as a primitive axiom of his formal theory was his inability to resolve the Caesar problem without appealing to value-ranges; in the last section, I argued that Frege ought to have concluded, on similar grounds, that the appeal to value-ranges failed to accomplish what was required of it—and so that he had no account of our apprehension of logical objects. One might wonder, however, whether this can be right. For one thing, Frege seems perfectly willing to accept Basic Law V as a primitive axiom, though he appears to have known that he could no more solve the version of the Caesar problem that arises in connection with it than he could solve the version that arises in connection with HP.

Frege's fondness for Basic Law V should not be overstated, however. He was famously dissatisfied with it even before Russell's discovery of the contradiction. He writes, in a famous passage from the Introduction to *Grundgesetze*:

> A dispute can arise, so far as I can see, only with regard to my basic law (V) concerning value-ranges, which logicians perhaps have not yet expressly enunciated, and yet is what people have in mind, for example, where they speak of the extensions of concepts. I hold that it is a law of pure logic. In any event the place is pointed out where the decision must be made. (Frege, 1962, v. I, p. vii)

But *what* decision must be made here? This remark is preceded by the following:

> Of course the pronouncement is often made that arithmetic is merely a more highly developed logic; yet that remains disputable so long as transitions occur in proofs that are not made according to acknowledged laws of logic, but seem rather to be based upon something known by intuition. Only if these transitions are split up into logically simple steps can we be persuaded that the root of the matter is logic alone. (Frege, 1962, v. I, p. vii)

It seems to me that the dispute Frege envisions is not a dispute about the *truth* of Basic Law V but rather one about its epistemological status: a dispute about whether it is a law of logic.

Showing that it is possible to derive arithmetic from certain axioms, whatever they may be, cannot decide the question of arithmetic's epistemological status on its own. The point should be obvious: If the laws of arithmetic follow logically from Basic Law V, then they are laws of logic if Law V is; if the laws of arithmetic follow logically from HP, then they are laws of logic if it is. A dispute can always arise, in principle, concerning the logical character of one's axioms and rules of inference, and Frege knew as much. His most transparent statement of this point is in his 1897 paper "On Mr. Peano's Conceptual Notation and My Own":

> I became aware of the need for a conceptual notation when I was looking for the fundamental principles or axioms upon which the whole of mathematics rests. *Only after this question is answered* can it be hoped to trace successfully the springs of knowledge upon which this science thrives. (Frege, 1984, op. 362, emphasis added)

If "the axioms upon which the whole of mathematics rests" were those of the formal theory of *Grundgesetze*, then the question of the epistemological status of Basic Law V would become the critical one, the one on which the epistemological status of arithmetic itself would turn. But, as Frege notes, "[a] dispute can arise" regarding it, a dispute he all but explicitly acknowledges he cannot resolve: Frege "hold[s] that it is a law of pure logic", but he has no convincing argument that it is.[13]

Frege was thus dissatisfied with Basic Law V for two sorts of reasons. On the one hand, he was unable to resolve the Caesar problem as it arose in connection with it; on the other, he had no defense of his claim that it is a law of logic. These two difficulties are not unrelated.

To understand the connection, it is useful to compare the case of the axioms of Euclidean geometry. Frege maintains that these axioms are non-logical truths because he holds that our knowledge of them depends upon intuition. More precisely, his view is that *apprehension of the objects of geometry* requires intuition of them and that our knowledge of the truth of the axioms is founded upon that intuition. Similarly, the question whether Basic Law V is a truth of logic is, for Frege, the question whether our recognition of its truth requires intuition or sense-experience. This question, in turn, Frege takes to reduce to the question how we apprehend the *objects* to which reference is made in Basic Law V: Only if we can apprehend value-ranges as

[13] For further discussion of the issues raised in the last two paragraphs, see my paper "Frege and Semantics" (Heck, 2010).

logical objects can we recognize the truth of Basic Law V independently of intuition and experience; only then can we recognize Basic Law V as a law of logic rather than as a law of one of the 'special sciences'.

In the case of HP, Frege's asking how one can apprehend numbers as logical objects is—or so I am suggesting—a way of asking how one can know HP to be true. If HP is to be a truth of logic, we must be able to recognize its truth without relying upon either experience or intuition, whence we must be able to apprehend the *references* of numerical terms (i.e., numbers) without perceiving or intuiting them.[14]

It is important to remember that Frege raises the Caesar problem in the context of a certain argument: It is easy to be distracted by its generality, to forget that it is not raised in a vacuum, as if Frege were asserting that it is a requirement on *any* acceptable definition of a singular term that it decide the truth-values of all identity statements containing that term.[15] The Caesar problem is raised as an objection to the claim that HP completely explains (identity statements involving) names of numbers. Now, again, the question under discussion at this point in *Die Grundlagen* is how we apprehend numbers as objects. So the view to which the Caesar problem is raised as an objection is this: We apprehend numbers as the referents of names of the form "the number of $F$s", and our understanding of these names consists entirely in our grasp of HP.[16]

Now, as has been said, it is broadly agreed that Frege's objection to this view is that

> [HP] will not, for instance, decide for us whether [Caesar] is the same as the [number zero]—if I may be forgiven an example which looks nonsensical. Naturally, no one is going to confuse [Caesar] with the [number zero]; but that is no thanks to our definition of [number]. (Frege, 1980a, §66; example changed)

But it is rarely mentioned, because it is not thought important, that Frege takes for granted that we *do* recognize that Caesar is not a number: Frege's objection is *not* that HP does not decide whether the singleton of the null set

[14] I am thus suggesting that Frege held that intuition in mathematics is primarily intuition of objects rather than intuition of truths. For the distinction, see Parsons's paper "Mathematical Intuition" (Parsons, 1980).

[15] This is a relatively common view of the Caesar problem. The demand that functions be defined for all arguments is, of course, characteristic of Frege's later writings. But I know of no real evidence that Frege held this view in *Die Grundlagen*, let alone evidence that the Caesar problem is simply a manifestation of this more general demand.

[16] This view, I take it, is similar to that defended by Wright in his book *Frege's Conception of Numbers as Objects* (Wright, 1983).

is the number zero (which, on Frege's explicit definition, it happens to be). The example Frege chooses is one about which he takes us to have strong intuitions: Whatever numbers may be, Caesar is not among them. Thus, there must be *more* to our apprehension of numbers than a mere recognition that they are objects that satisfy HP. Something explains *why* "no one is going to confuse [Caesar] with the [number zero]". Frege is thus insisting that any complete account of our apprehension of numbers as objects must include an account of how we recognize that Caesar is not a number. But HP alone yields no such explanation.[17]

The Caesar problem is not intended to show only that our apprehension of numbers as *logical* objects cannot be explained in terms of our knowledge that they satisfy HP. Frege's discussion of the Caesar problem takes place, after all, in the context of a discussion of names of directions, and directions are surely not logical objects. The intended lesson of the Caesar problem, in the case of directions, therefore cannot be that our knowledge that directions satisfy the appropriate abstraction principle does not explain our apprehension of directions as *logical* objects. The lesson is supposed to be that our apprehension of directions *as objects at all* cannot be explained in terms of our knowledge that they satisfy the abstraction principle. For again: If we apprehended directions only by recognizing them to satisfy the appropriate abstraction principle, we would have no basis on which to claim that England is not the direction of the Earth's axis; but we all *do* recognize that England is not the direction of the Earth's axis, so there must be something about our apprehension of directions, something about our capacity to refer to them, that is not captured by the abstraction principle.

Because the Caesar problem, as here interpreted, concerns only mixed identity statements about which we have reasonably strong intuitions, it may not be the same problem as the one Frege raises in *Grundgesetze* §10, which concerns all sentences (of the formalism) in which value-range terms occur. To solve the Caesar problem in the form in which it is raised in *Die Grundlagen*, we must explain in virtue of what we recognize (e.g.) that Caesar is not a number. It is not obvious that doing so requires that we fix the truth-values of all mixed identities.

Moreover, even if Frege had been able to fix the truth-values of all mixed identity statements, not just any way of doing so would have solved the Caesar problem to his satisfaction. To see this, note that the truth-values of

[17] It is just this that Wright (1983, §xiv) proposes to deny: He thinks that HP does, in some way, decide whether Caesar is a number. See also the later discussion by Hale and Wright (2001).

all such sentences could be fixed, in principle, by identifying numbers with *non*-logical objects. If we suppose, for the moment, that the only (cardinal) numbers are countable numbers, numbers may be identified with numerals. And if we take the same liberties that Frege did and suppose that it is already known what numerals are—that is, if we suppose we already know such things as whether Caesar is a numeral and, if so, which one he is—then this stipulation will decide the truth-values of mixed identity statements. Caesar, for instance, not being a numeral, is not a number, either.

Frege surely would not have accepted such a solution to the Caesar problem. Why not? Why should the identification of numbers with non-logical objects pose any threat to Frege's logicism? It is far from clear that an identification of numbers with numerals *need* pose any real threat to logicism. One might argue, for example, that the Caesar problem does not raise any questions about the truth of, or grounds for, HP: Its truth is established prior to, or independently of, any such identification, which serves only to fix the truth-values of certain statements (for whatever reason one might want to do that).[18] On this view, we apprehend numbers as the objects of which HP is true. HP, in turn, we know to be true independently of any intuition or experience because, say, we recognize it to be analytic of the concept of cardinal number. Thus, we apprehend numbers as logical objects because our recognition of the truth of HP requires neither intuition nor experience. Note the order of explanation: It is essential to the argument that our knowledge that HP is true does not depend upon any prior apprehension of numbers. So, to summarize: On this view, we apprehend numbers as the objects of which HP is true; if numbers are numerals and knowledge of HP requires neither intuition, nor experience, nor even *thought about* numbers, we can apprehend *numerals* as logical objects. That may be surprising, but no absurdity appears to be forthcoming: As Frege emphasizes time and again, objects may be given in different ways.

Why, then, would Frege have rejected the identification of numbers as numerals? What one would like to say is that, if numbers are numerals, our knowledge about numbers—in particular, our knowledge of HP and its consequences—would be corrupted by our knowledge of the relevant facts about numerals. Now, granted, *some* of our knowledge about numbers would not be logical knowledge; for example, our knowledge that the number zero is round would not be logical in character. But there is no obvious reason that *all* one's knowledge about numbers would be so 'corrupted', and we have seen that it is possible to reject this claim in a principled fashion, by

[18] There is a bit more on this matter in Chapter **??**.

maintaining that our apprehension of the truth of HP does not require any prior apprehension of numbers themselves. I suggest, however, that for Frege the identification of numbers as objects of another kind (whether numerals, as in the example, or value-ranges) is to be part of an *explanation* of the truth of HP, part of an account of how it can be known to be true, and so part of an explanation of its epistemological status. If, with this in mind, we re-describe what the identification of numbers as numerals accomplishes, we see immediately why Frege would have rejected it: If numbers are identified with numerals, and if the truth of HP is explained in terms of the existence of a mapping from concepts to numerals, then our recognition of the truth of HP, so understood, depends upon our recognition that there is such a mapping and so that there are enough numerals to constitute the range of such a mapping (at least countably many). Such an explanation of the truth of HP would presumably not show it to be a truth of logic: It would, rather, show it to be a truth of whatever sort the relevant truths about numerals are; the epistemological status of HP would then depend upon the epistemological status of the relevant knowledge about numerals.[19]

Frege is thus maintaining that our knowledge of the truth of HP depends upon our knowledge about the objects with which numbers are identified. To put the point in more Fregean language, his view is not that we apprehend numbers as logical objects because we recognize HP to be a truth of logic; his view is that we recognize HP to be a truth of logic because we apprehend numbers as logical objects and recognize HP to be true of them.[20]

That this is Frege's view is clear from his discussion, in *Grundgesetze*, of explanations of terms by means of abstraction principles (for short, 'contextual explanations'). Regarding such explanations, Frege writes:

> . . . [W]e may not define a symbol or word by defining an expression in which it occurs, whose remaining parts are known. For it would first be necessary to investigate whether—to use a readily understandable metaphor from algebra—the equation can be solved for the unknown, and whether the unknown is unambiguously determined. (Frege, 1962, v. II, §66)

If we were to take HP as a contextual explanation of names of numbers, two

[19] For present purposes, it matters little what the epistemological status of our knowledge about numerals might be, other than that the relevant truths about numerals are not logical. But Parsons (1980) argues that numerals, as types, are objects of intuition.

[20] There are strong similarities between the point now being argued and Michael Dummett's view that terms explained by means of abstraction principles refer, though not in a manner suitable to a realist interpretation of them (Dummett, 1981, ch. 14).

questions would arise: Whether there are *any* objects that satisfy HP, and whether there is any *unique* set of objects that satisfy it.

The latter problem is the focus of Frege's attention in *Die Grundlagen*, the question being, as it were, *which* objects the numbers are. The former problem arises naturally, however, from reflection on the Caesar problem. As said earlier, the Caesar problem is intended to show that we cannot explain our capacity to refer to numbers (to apprehend numbers as objects) solely in terms of our knowledge that they satisfy HP: If we take HP as a contextual explanation of names of numbers, we have no explanation of how we apprehend numbers as objects at all. But, if not, we are presumably without any defense of the claim that there are such objects.[21] The question to which the Caesar problem leads is thus: What distinguishes HP, if taken as a primitive logical law, from a 'creative definition', where to give a creative definition of an object is to specify an equation it must satisfy and then to stipulate that there is to be an object satisfying that equation (Frege, 1962, v. II, §§143–4)?[22] By the time he wrote *Grundgesetze*, at least, Frege not only thought he could not answer this question, he thought it unanswerable: HP, construed as a contextual explanation, simply is a creative definition.

Now, as I have emphasized, Frege was also without a solution to the Caesar problem as it arises in the case of value-ranges. Indeed, his claim that the Caesar problem, in general, can be satisfactorily resolved only by identifying numbers (or directions, or what have you) with value-ranges threatens to make it impossible to resolve the Caesar problem, as it arises for value-ranges *themselves*: It is obviously useless to attempt an identification of value-ranges with value-ranges, and Frege has foresworn any other sort of solution. The question therefore arises whether Basic Law V is not *itself* a creative definition:

> . . . [S]omebody might indicate that we ourselves have nevertheless constructed new objects, viz. value-ranges (vol. I, §§3, 9, 10). What, then, did we do there? or, rather, in the first place, what did we not do? We did not enumerate properties and then say: we construct a thing that is to have these properties. (Frege, 1962, v. II, §146)

What is important here is Frege's conception of how Basic Law V differs

[21] One could avoid this conclusion, as said above, if one denied—as I take it Wright does—the implicit assumption that the explanation of our knowledge of HP depends upon an explanation of our apprehension of numbers as objects, rather than conversely.

[22] The discussion of 'postulationism' in §§92ff of *Die Grundlagen* is remarkably similar, both in general spirit and, at times, in detail to the cited discussion in *Grundgesetze*.

from a creative definition. First, he insists that he is *not* just stipulating that there are to be objects that satisfy Basic Law V. Then he argues that what he really said—in *Grundgesetze* §3, quoted above—was:

> If a (first-level) function (of one argument) and another function are such as always to have the same value for the same argument, then we may say instead that the value-range of the first is the same as that of the second. We are then *recognizing something common* to the two functions, and we call this the value-range of the first function and also the value-range of the second function. (Frege, 1962, v. II, §146, my emphasis)

That is: In making the transformation from the thought that $\forall x(Fx = Gx)$ to the thought that $\grave{\epsilon}(F\epsilon) = \grave{\epsilon}(G\epsilon)$, we are "recognizing something common" to the functions $F\xi$ and $G\xi$. Now, it is extremely tempting to read Frege's words "we may say instead" as suggesting that this transition is merely verbal—or, perhaps, merely conceptual—and so that we require no justification to make that transition. But we have already seen that Frege must deny that our recognition (or apprehension) of value-ranges can be explained in terms of our recognition that they satisfy Law V: The Caesar problem prevents explanations of that form. The explanation must instead proceed the other way:

> We must regard it as a fundamental law of logic that we are justified *in thus recognizing something common to both*, and that *accordingly* we may transform an equality holding generally into an equation (identity). (Frege, 1962, v. II, §146, my emphasis)

Thus, it is to be a law of logic[23] that we may *recognize* something common to co-extensional functions: It is *because* we can so recognize (apprehend) value-ranges that the inference from the co-extensionality of functions to the identity of their value-ranges is permitted, which inference is then formalized in Basic Law V. Our recognition of the truth of Law V is thus to be grounded in our apprehension of value-ranges as objects: Its status as a law of logic therefore requires that we be able to apprehend value-ranges as logical objects, that is, apprehend them without relying upon intuition or experience.

But, as we have seen, Frege has no argument that value-ranges are logical objects. And he has no argument that there are any such objects.

[23] Frege would seem to be using the word "logic" here in the broad sense in which it was used in German philosophy in the nineteenth century, so that it included parts of what we would now call epistemology and metaphysics.

## 4 Closing

Why was Frege unwilling to abandon Basic Law V, install HP as a primitive axiom, and derive the laws of arithmetic from it? The answer at which we have arrived is this: Without the explicit definition of numbers in terms of extensions, Frege could not solve the Caesar problem as it arises for numbers; he could not explain how we apprehend numbers as objects—logical or otherwise—and so could not explain how we know HP to be true. On the other hand, however, Frege was also without a solution to the Caesar problem as it arises for value-ranges: He was unable to explain how we apprehend value-ranges as objects—logical or otherwise—and so was unable to explain how we know Basic Law V to be true. What we have seen is thus that the objections to treating HP as a primitive axiom can also be raised against Frege's treatment of Basic Law V as a primitive axiom: The situations seem exactly parallel, and Frege regarded them as parallel. Why then was he willing to accept Basic Law V but not HP as a primitive axiom?

To this question, there is a dissatisfyingly simple answer. Though he has no argument that extensions are logical objects, Frege does not expect to meet much opposition on this point. Most of his admissions that he is unable to produce such an argument are immediately followed by remarks like the following:

> Logicians have long since spoken of the extension of a concept, and mathematicians have used the terms set, class, manifold; what lies behind this is a similar transformation [from the generality of an identity to an identity of value-ranges]; for we may well suppose that what mathematicians call a set (etc.) is nothing other than the extension of a concept, even if they have not always been clearly aware of this. (Frege, 1962, v. II, §147; see also v. I, p. vii)

Dialectically, the supposition that there are value-ranges, and that they satisfy Basic Law V, was not an unreasonable one for Frege to make.

The dialectical situation with respect to HP, however, could not be more different. To suggest that we regard it as a fundamental law of logic that we are justified in recognizing something common to two *equinumerous* concepts, and that accordingly logic allows us to transform a statement of equinumerosity into an identity of numbers (see Frege, 1962, v. II, §146, again), would blatantly beg the question whether arithmetic is a branch of logic. To derive arithmetic from Basic Law V, and then to suggest that Law V be taken as a fundamental law of logic, is to make substantial dialectical progress, progress rightly characterized as showing "where the decision must be made" (Frege, 1962, v. I, p. vii). To derive arithmetic from HP, and then merely to remark that *it* must be regarded as a fundamental law of logic,[24] is to make no such progress.

That does not imply that HP is not *in fact* the right place for the decision to be made.

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[24] Let me make it clear that contemporary neo-logicists do not merely make such a remark.

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